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# Report for Master's degree internship at ESA's Advanced Concepts Team

SUBJECT: ANALYTICAL SOLUTIONS TO THE PROBLEM OF POSITION DETERMINATION IN THE RPS FRAMEWORK USING POST-MINKOWSKIAN FORMALISM

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# 1 Introduction

A satellite from a global navigation satellite system (GNSS) operates by broadcasting signals in which the time indicated by its on-board clock, known as its *proper time*, is encoded. On the user's end, a GNSS receiver compares the time of reception of signals from at least four satellites with it's own time in order to determinate the position of the user.

Let us note $t_e$ the time of emission of such a signal, and $t_r$ its time of reception. When this receiver picks up a signal from a satellite, it can infer that it lies on a sphere of radius $\frac{|t_e - t_r|}{c}$ centred around the satellite. Performing this measure with signals coming from three satellites, and knowing their localisation, the receiver can in principle determine the three coordinates $\{x, y, z\}$ of his position[(i)]. In fact, the receiver's clock will often have poor accuracy, so the reception time $t_r$ can be treated as an unknown as well [1], and signals from a fourth satellite will be needed.

If we call $\{t, x, y, z\}$ the receiver's coordinates, the problem of determining the its position reduces to solve the following system of equations [1] :

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = c^2(t - t_i)^2, \qquad i = 1, 2, 3, 4, \tag{1}$$

with $i$ running over the satellites.

Present-day GNSS's, such as GPS, GLONASS or Galileo, rely on the Newtonian concept of absolute space and time and make use of correction terms in order to account for the effects of relativity. They also rely on ground station to localise and synchronise the satellites of the GNSS constellation. A more natural point of view would be to directly consider the "actual" spacetime, as described by Einstein's theory of general relativity, so that all computations are correct by default, with no need for additional corrections [2][3].

In order to implement such a fully-relativistic GNSS, which we will call from now on "RPS" for *Relativistic Positioning System*, one has to be able to compute the time of flight of photons in a curved spacetime, which is not a trivial problem. The outcome of this internship was the discovery of analytical solutions to the problem of position determination for circular orbits in a Schwarzschild spacetime and minimised numerical calculations for elliptical orbits; these come in the form of series expansions.

# 2 Theoretical elements

## 2.1 Post-Minkowskian metric

The assumptions about the spacetime we'll be considering are the same as in [4]. We will recall them here to have a self-contained discussion.

We will be using a static, spherically symmetric metric which can be thought as the post-Minkowskian generalisation of the Schwarzschild metric. We will also suppose that there is a region $D_h$ where the metric is smooth and asymptotically flat, and that can be thought as the gravitational field of a spherical body of mass $M$. In practice this region will be the outside of a sphere of radius $r_h = \frac{GM}{2c^2}$. Throughout this paper, all geodesic paths considered will remain

(i) Actually, there are two solutions to this problem, and the receiver will have to select the solution which is closer to the earth's surface [1].

within $D_h$. In other words, we're assuming that all events take place outside the Schwarzschild radius of the central body, Earth, which is largely justified for RPS calculations.

We're using the same notation as in [4], namely:

$$ds^2 = \mathcal{A}(r)\big(dx^0\big)^2 - \mathcal{B}^{-1}\delta_{ij}dx^i dx^j, \tag{2}$$

where, in spherical coordinates, $\delta_{ij}dx^i dx^j = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ and with $\mathcal{A}(r)$ and $\mathcal{B}^{-1}(r)$ given by [4]

$$\mathcal{A}(r) = 1 - 2\frac{m}{r} + 2\beta\frac{m^2}{r^2} - \frac{3}{2}\beta_3\frac{m^3}{r^3} + \beta^4\frac{m^4}{r^4} + \sum_{n=5}^{\infty}\frac{(-1)^n n}{2^{n-2}}\beta^n\frac{m^n}{r^n}, \tag{3}$$

$$\mathcal{B}^{-1}(r) = 1 + 2\gamma\frac{m}{r} + \frac{3}{2}\epsilon\frac{m^2}{r^2} + \frac{1}{2}\gamma_3\frac{m^3}{r^3} + \frac{1}{16}\gamma_4\frac{m^4}{r^4} + \sum_{n=5}^{\infty}(\gamma_n - 1)\frac{m^n}{r^n}, \tag{4}$$

in terms of the generalised post-Newtonian parameters, which are all equal to 1 in general relativity. By setting $\mathcal{U}(r) = \frac{1}{\mathcal{A}(r)\mathcal{B}(r)}$, we can define a new metric $d\tilde{s}$ of the form:

$$d\tilde{s}^2 = \big(dx^0\big)^2 - \mathcal{U}(r)\big(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\big), \tag{5}$$

where the potential $\mathcal{U}(r)$ may be written as

$$\mathcal{U}(r) = 1 + 2(1+\gamma)\frac{m}{r} + \sum_{n=2}^{\infty} 2\kappa_n\frac{m^n}{r^n}, \tag{6}$$

with $m \equiv GM/c^2$ and the $\kappa_n$ being constants related to the post-Newtonian coefficients, which are calculated directly using formulae (3), (4), and (6). Explicitly,

$$\kappa = 2(1+\gamma) - \beta + \frac{3}{4}\epsilon, \tag{7}$$

$$\kappa_2 = \kappa, \tag{8}$$

$$\kappa_3 = 2\kappa - 2\beta(1+\gamma) + \frac{3}{4}\beta_3 + \frac{1}{4}\gamma_3, \tag{9}$$

$$\kappa_4 = 8(1+\gamma) + 2\beta^2 - \frac{1}{2}\beta(16\gamma + 3(8+\epsilon)) + \frac{3}{2}(2+\gamma)\beta_3 - \frac{\beta_4}{2} + \frac{\gamma_3}{2} + \frac{\gamma_4}{32} + 3\epsilon, \tag{10}$$

so that in general relativity, one has

$$\kappa = \kappa_2 = \frac{15}{4}, \quad \kappa_3 = \frac{3}{2}, \quad \kappa_4 = \frac{129}{32}. \tag{11}$$

This new metric has the same geodesics as $ds$.

### 2.1.1 Quasi-Minkowskian light rays

Borrowing another definition from [4], we will only consider paths traveled by light rays which are entirely confined to $D_h$ and which can be parametrised as

$$\begin{cases} x^0 = ct_a + \xi|\mathbf{x}_B - \mathbf{x}_A| + \sum_{n=1}^{\infty} X^0_{(n)}(\mathbf{x}_A, \mathbf{x}_B, \xi) \\ \mathbf{x} = \mathbf{z}(\xi) + \sum_{n=1}^{\infty} \mathbf{X}_{(n)}(\mathbf{x}_A, \mathbf{x}_B, \xi) \end{cases}, \tag{12}$$

where $\xi$ is an affine parameter going from 0 to 1, $\mathbf{z}(\xi) = \mathbf{x}_A + \xi(\mathbf{x}_B - \mathbf{x}_A)$ is the straight line joining $\mathbf{x}_A$ and $\mathbf{x}_B$, and $X_{(n)}$ is the $n$-th correction to the path with vanishing boundary conditions. These types of geodesics will be referred to as *quasi-Minkowsian*.

Intuitively, asking that all light rays follow quasi-Minkowskian paths will exclude "extreme" relativistic paths with very strong light deflections, such as the ones seen near black holes, which is perfectly fine for our application to RPS.

## 2.2 Time transfer function

Let us consider a light ray emitted at $x_A$ and received at $x_B$. The time transfer function is defined as the coordinate travel time of a light ray connecting the emission point $x_A$ and the reception point $x_B$ [5], which is either a function of $\mathbf{x}_A$, $\mathbf{x}_B$ and $t_A$, or of $\mathbf{x}_A$, $\mathbf{x}_B$ and $t_B$. We can thus define two time transfer functions $\mathcal{T}_e$ and $\mathcal{T}_r$ as follows [5] :

$$\boxed{\frac{1}{c}\int_{x_B}^{x_A} dx^0 = t_B - t_A = \mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = \mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B)}. \tag{13}$$

Note that if the metric is static, these two functions are the same, as the dependence in the emission/reception time vanishes. This is indeed the case for our metric, so from now on we shall simply note the time transfer function $\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)$.

Let us note $\mu$ the cosine of the angle formed by the two vectors $\mathbf{x}_A$ and $\mathbf{x}_B$ :

$$\mu = \mathbf{n}_A \cdot \mathbf{n}_B = \cos(\phi_A - \phi_B), \tag{14}$$

where $\mathbf{n}_i = \mathbf{x}_i/r_i$, $r_i = |\mathbf{x}_i|$, with $i = A, B$. With this notation, we can consider the time transfer function to be a function of $r_A$, $r_B$ and $\mu$ :

$$\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B) = \mathcal{T}(r_A, r_B, \mu). \tag{15}$$

It results that, under the assumptions required for (12) to be true, the time transfer function can be expanded as follows [4]:

$$\mathcal{T}(r_A, r_B, \mu) = \frac{R_{AB}}{c} + \sum_{n=1}^{\infty} \mathcal{T}^{(n)}(r_A, r_B, \mu), \tag{16}$$

where

$$R_{AB} = \sqrt{r_A^2 + r_B^2 - 2 r_A r_B \cos\mu}, \tag{17}$$

the zeroth-order term of the expansion (16), is the Euclidean distance between $\mathbf{x}_A$ and $\mathbf{x}_B$.

As for the next terms, we built on the expansion scheme presented in [4] to compute the expression (16) up to the fourth order :

$$\mathcal{T}^{(1)}(r_A, r_B, \mu) = m\frac{1+\gamma}{c} \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right), \tag{18}$$

$$\mathcal{T}^{(2)}(r_A, r_B, \mu) = \frac{m^2}{c}\frac{R_{AB}}{r_A r_B}\left[\kappa_2 \frac{\arccos\mu}{\sqrt{1-\mu^2}} - \frac{(1+\gamma)^2}{1+\mu}\right], \tag{19}$$

$$\mathcal{T}^{(3)}(r_A, r_B, \mu) = \frac{m^3}{c}\frac{R_{AB}(r_A + r_B)}{r_A^2 r_B^2 (1+\mu)}\left[\kappa_3 - (1+\gamma)\kappa_2 \frac{\arccos\mu}{\sqrt{1-\mu^2}} + \frac{(1+\gamma)^3}{1+\mu}\right], \tag{20}$$

$$
\begin{aligned}
\mathcal{T}^{(4)}(r_A, r_B, \mu) = & \frac{m^4}{c} \frac{R_{AB}}{r_A^3 r_B^3 \left(1-\mu^2\right)^2} \Bigg[ -(1+\gamma)^4 \frac{5\left(2r_A^2+2r_B^2+r_A r_B(3-\mu)\right)(1-\mu)^2}{6(1+\mu)} \\
& + \kappa_2 (1+\gamma)^2 \frac{\left(2r_A^2+2r_B^2+r_A r_B(3-\mu)\right)(1-\mu)^2 \arccos\mu}{\sqrt{1-\mu^2}} \\
& - \kappa_2^2 \frac{\arccos\mu \left(R_{AB}^2\left(1-\mu^2\right) - (r_B - r_A\mu)(r_B\mu - r_A)\sqrt{1-\mu^2}\arccos\mu\right)}{2\sqrt{1-\mu^2}} \\
& + (1+\gamma)\kappa_3 \left((1-\mu)\left(\left(r_A^2+r_B^2\right)(3-\mu) + 2r_A r_B(1-3\mu)\right) + R_{AB}^2\sqrt{1-\mu^2}\arccos\mu\right) \\
& + \frac{\kappa_4}{2}\left(\left(2r_A r_B - \left(r_A^2+r_B^2\right)\mu\right)\left(1-\mu^2\right) + R_{AB}^2\sqrt{1-\mu^2}\arccos\mu\right) \Bigg]. \qquad (21)
\end{aligned}
$$

The fourth-order term is, as far as we know, new.

## 2.3 Equation of motion of the satellites

We will identify the event A with the 4-coordinates of a satellite when it emits a signal and event B with the 4-coordinates of the receiver when it receives it. Since the defining equation of the time transfer function (13), which we will refer to as the *time transfer equation*, establishes a link between all of them, we will be able to retrieve any of these coordinates from the others. By having several such equations, it is possible to determine all the four coordinates of the receiving event, which is the point of a RPS.

To that end, we will need the solutions of the equations of motion of the satellites, which are considered here to be finite-mass objects in free-fall in a Schwarzschild spacetime. In general, these solutions will be functions of $\nu \equiv r_S/r_0$, the Schwarzschild radius of the central body divided by the characteristic radius of the orbits (a constant of motion which, for circular orbits, coincides with the radius of the orbit itself). Since this quantity is expected to be very small, we will be able to expand these solutions in a power series of $r_S/r_0$, and we shall present them in their expanded form directly :

### 2.3.1 Circular equatorial orbits

The dynamics of such an object is very well-known [6] :

$$
\begin{aligned}
t(\tau) &= t_0 + \tau\left[1 + \frac{3}{4}\nu + \mathcal{O}\left(\nu^2\right)\right], \\
r(\tau) &= r_0, \\
\theta(\tau) &= \frac{\pi}{2}, \\
\phi(\tau) &= \phi_0 + \tau\left[\frac{c}{r_0}\sqrt{\frac{\nu}{2}} + \mathcal{O}\left(\nu^2\right)\right],
\end{aligned}
\qquad (22)
$$

where $\tau$ is of course the proper time of the satellite.

### 2.3.2 Elliptical equatorial orbits

The solutions to the equations of movement take a more complicated form, notably because they can't be expressed as functions of the proper time anymore [7]. Let's first note

$$Y = 2\pi r_0 \nu^{-1/2}(1-e^2)^{-3/2}. \tag{23}$$

We then have [7]:

$$\begin{aligned}
t(\psi) &= t_0 + \frac{Y}{2\pi}\left[(\psi - \sin\psi) + \frac{\nu}{2}\left(1-e^2\right)(3\psi - e\sin\psi) + \mathcal{O}\left(\nu^2\right)\right],\\
r(\chi) &= \frac{r_0}{1+e\cos\chi},\\
\theta(\chi) &= \frac{\pi}{2},\\
\phi(\chi) &= \chi + \nu(3\chi + e\sin\chi) + \mathcal{O}\left(\nu^2\right),\\
\tau(\psi) &= \frac{Y}{2\pi}\left[(\psi - \sin\psi) + 3\nu(1-e^2)\psi + \mathcal{O}\left(\nu^2\right)\right],
\end{aligned} \tag{24}$$

where $\chi$ the relativistic anomaly of the satellite and $\psi$ its eccentric anomaly. These two parameters are related to each other via

$$(1+e\cos\chi)(1-e\cos\psi) = 1-e^2. \tag{25}$$

### 2.3.3 Inclined orbits

To convert any of these equatorial orbits into an inclined orbits, we just apply quadri-dimensional rotations matrices to the position vector $(t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$ of the satellite, where $\lambda$ is the variable used to parametrise the orbits (*i.e.* either $\tau$, $\psi$ or $\chi$). Calling $\alpha$, $\beta$ and $\gamma$ the rotation angle with respect to the $x$-, $y$- and $z$-axis respectively, it can be shown with elementary algebra that the new position vector $(T(\lambda), R(\lambda), \Theta(\lambda), \Phi(\lambda))$ is given by :

$$\begin{aligned}
T(\tau) &= t(\lambda)\\
R(\tau) &= r(\lambda)\\
\Theta(\lambda) &= \arccos\left(\frac{\sin\beta\cos\phi(\lambda) + \sin\alpha\cos\beta\sin\phi(\lambda)}{\sqrt{1+\sin\alpha\sin 2\beta\sin 2\phi(\lambda)}}\right)\\
\Phi(\lambda) &= \arctan\left(\frac{\cos\beta\cos\phi(\lambda)\sin\gamma - \cos\alpha\cos\gamma\sin\phi(\lambda) + \sin\alpha\sin\beta\sin\gamma\sin\phi(\lambda)}{\cos\beta\cos\gamma\cos\phi(\lambda) + \cos\gamma\sin\alpha\sin\beta\sin\phi(\lambda) + \cos\alpha\sin\gamma\sin\phi(\lambda)}\right)
\end{aligned} \tag{26}$$

# 3 Method towards the solution

We now have all the elements at hand to solve the time transfer equation, with event A being the emitting satellite and event B being the receiving end. Let us rewrite it accordingly :

$$\boxed{t_B - t_A(\lambda_A) = R_{AB}(r_A(\lambda_A), r_B, \mu(\lambda_A)) + \sum_{n=1}^{order} \mathcal{T}^{(n)}(r_A(\lambda_A), r_B, \mu(\lambda_A)),} \tag{27}$$

where we adjoined a subscript $A$ to the parameter $\lambda$ to emphasise the fact that it belongs to the satellite.

Our goal is to find the coordinates of $x_B = (t_B, r_B, \theta_B, \phi_B)$ in an analytical (expanded) form. As explained earlier, determining the four coordinates of the receiving point will require signals from four satellites, which translates here as having a system of four equations such as (27), each with a different $\lambda_A$.

Once we have these four equations, we then postulate that the unknowns can be written on an expanded form :

$$\begin{aligned} t_{B,exp} &= t_B^{(0)} + \sqrt{\nu} t_B^{(1/2)} + \nu t_B^{(1)} + ... + \nu^{order} t_B^{(order)}, \\ r_{B,exp} &= r_B^{(0)} + \sqrt{\nu} r_B^{(1/2)} + \nu r_B^{(1)} + ... + \nu^{order} r_B^{(order)}, \\ \theta_{B,exp} &= \theta_B^{(0)} + \sqrt{\nu} \theta_B^{(1/2)} + \nu \theta_B^{(1)} + ... + \nu^{order} \theta_B^{(order)} \\ \phi_{B,exp} &= \phi_B^{(0)} + \sqrt{\nu} \phi_B^{(1/2)} + \nu \phi_B^{(1)} + ... + \nu^{order} \phi_B^{(order)}, \end{aligned} \tag{28}$$

By injecting these formulae into our system, we were able to solve it analytically order by order. And by adding all the orders as specified in eqs. (28), we reconstruct the solutions. We carried out this procedure up to the fourth order. The solutions obtained are too bulky to be presented here, but will be made available online soon. The zeroth-order contribution of each coordinate gave the position of the receiver in flat space-time, as expected.

# 4 Results on simulated data

We tested the consistency of our solutions by using them in simulated satellite constellations. We proceeded as follows :

1. Select the initial phase and orbital parameter for each satellite, and select the $(t_B, r_B, \theta_B, \phi_B)$ coordinates of our target point.

2. Compute the proper times $\tau_1, \tau_2, ...$ that the satellites would have when they emit their signal so that they are all received at point $B$ at the time $t_B$ selected. This is done by solving the time transfer equation between these two points. We proceeded analytically for circular orbits (the only case when this was possible), by supposing the proper time takes on an expanded form, $\tau_i = \tau_i^{(0)} + \sqrt{\nu}\tau_i^{(1/2)} + \nu\tau_i^{(1)} + ... + \nu^{order}\tau_i^{(order)}$ and then solving the time transfer equation by orders, but a numerical solving algorithm was required for elliptical orbits. Note that this is of no concern, since we only simulate the proper times of the satellites here. In real life, these are simply broadcast by the satellites and decoded by the receiver. We also tested the results of both ways of solving against a known setup [2] and results agreed.

3. We then simply inject these four proper times into the solutions we found with the method described in section 5.3, along with the positions of the satellites (which are also known). If these solutions are correct, we should retrieve the coordinates chosen at step 1.

The method did work for all settings considered, which we will present now.

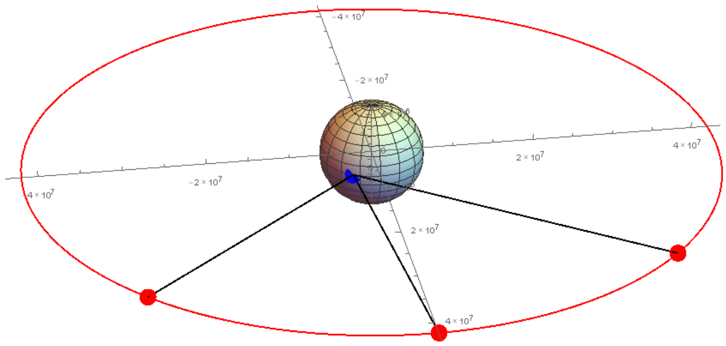


*Figure 1: Setting used to test the position solution with equatorial circular orbits. The blue dot represents the target point $B$. The red dots and the red circle respectively represent the satellites and their common orbit.*

## 4.1 Equatorial circular orbits.

We tried our method with the target point

$$x_B = (1\,\text{s}, 6.3 \times 10^6\,\text{m}, \pi/2\,\text{rad}, -\pi/6\,\text{rad}), \tag{29}$$

and a constellation of satellite obeying equations (22) with the following constants of motion

$$\begin{aligned} t_{0,i} &= 0\,\text{s} \quad \forall i, \\ r_{0,i} &= 4.2 \times 10^7\,\text{m} \quad \forall i, \\ \phi_{0,1} &= 0\,\text{rad}, \quad \phi_{0,2} = \pi/4\,\text{rad}, \quad \phi_{0,3} = -\pi/4\,\text{rad} \end{aligned} \tag{30}$$

The setting is illustrated in figure 4. Note that we only have three satellites and not four ; this is because, when we confine ourselves to the equatorial plane, the $\theta$ coordinate is fixed at $\pi/2$ for all objects, and thus needs not to be solved for.

We then simulated the proper times of each of the satellites using the time transfer equation up to the fourth order. All values here are presented with 30 digits. We realise it is much more than what will ever be useful for all practical purposes, but trying the method at such high precision is a good test of its robustness. We obtained :

| Sat. | $\tau$ [s] |
|---|---|
| 1 | 0.877649417616130052253210684004 |
| 2 | 0.863819405261826444311422542536 |
| 3 | 0.880078571445747170606343452927 |

Finally we retrieved the coordinates $t_B$, $r_B$ and $\phi_B$ of the target point by injecting the previous data into our solutions. The first column indicates the order up to which the calculation was performed, the second one the actual value obtained for that coordinates and the third represents the relative error with the expected value (29). The results for time of reception (expected answer : $t_B = 1$ second) were :

| Order | $t_B$ [s] | Error |
|---|---|---|
| 0 | 1.0000000311605433179589318339335 | $3.1161 \times 10^{-8}$ |
| 0.5 | 0.9999999998017222080340422394O5 | $1.9828 \times 10^{-10}$ |
| 1 | 1.0000000000000000668054914406170 | $6.6805 \times 10^{-17}$ |
| 1.5 | 1.0000000000000000002288630727880 | $2.2886 \times 10^{-19}$ |
| 2 | 0.9999999999999999999999997180360 | $2.8196 \times 10^{-25}$ |
| 2.5 | 1.0000000000000000000000000001330 | $1.3263 \times 10^{-28}$ |
| 3 | 1.0000000000000000000000000000000 | $9.4083 \times 10^{-34}$ |

We can see that we already reach a relative error of only $10^{-33}$ (and thus an absolute error of $10^{-33}$ seconds in this case, as $t_P = 1$ s) with the third order contribution. We thus felt it unnecessary to carry out the procedure up to higher orders in that situation. For the radius (expected answer : $r_B = 6.3 \times 10^6$ metres), we obtained :

| Order | $r_B$ [$10^6$ m] | Error |
|---|---|---|
| 0 | 6.2999898821799415766783556357470 | $-1.6060 \times 10^{-6}$ |
| 0.5 | 6.3000000009357758738224184546870 | $1.4854 \times 10^{-10}$ |
| 1 | 6.2999999999999937581822539969450 | $9.9076 \times 10^{-16}$ |
| 1.5 | 6.2999999999999999430350474651080 | $9.0421 \times 10^{-18}$ |
| 2 | 6.3000000000000000000000250689410 | $3.9792 \times 10^{-24}$ |
| 2.5 | 6.2999999999999999999999999735360 | $4.2006 \times 10^{-27}$ |
| 3 | 6.3000000000000000000000000000000 | $4.7364 \times 10^{-32}$ |

The remarks that can be made from here are similar as for the time of reception. Laslty, the results for the azimuthal angle (expected answer : $\phi_B = -\pi/6$ radians) :

| Order | $\phi_B$ [rad] | Error |
|---|---|---|
| 0 | -0.52366345680129748231908317295200 | $1.1235 \times 10^{-4}$ |
| 0.5 | -0.52359877556088578912946948705300 | $-7.1454 \times 10^{-11}$ |
| 1 | -0.52359877559830932873408237160600 | $1.9969 \times 10^{-14}$ |
| 1.5 | -0.52359877559829888924811591737900 | $3.0884 \times 10^{-17}$ |
| 2 | -0.52359877559829887307705249650700 | $1.0453 \times 10^{-22}$ |
| 2.5 | -0.52359877559829887307710722378900 | $1.2906 \times 10^{-26}$ |
| 3 | -0.52359877559829887307710723054700 | $1.6206 \times 10^{-32}$ |

For which the same remarks can be drawn. Overall, the radius is the only dimensionful coordinate of the receiver's position. The absolute error on this coordinate is $2.98 \times 10^{-25}$ metres, and this value can be taken as a rough estimation of the absolute error on the position in Cartesian coordinates.

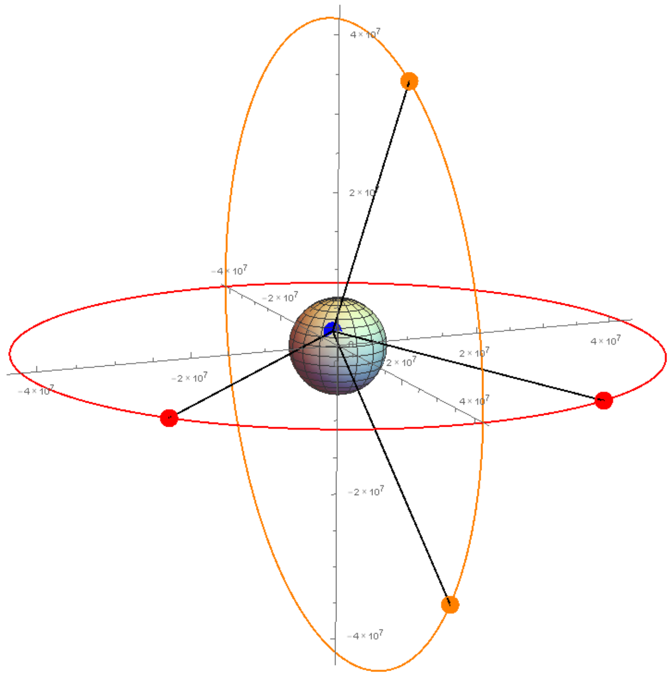


*Figure 2: Setting used to test the position solutions with equatorial circular orbits. The blue dot represents the target point $B$. The red dots and the red circle respectively represent the two satellites on the $\alpha = 0$ orbit, while the orange object represent the $\alpha = 90^\circ$ orbit.*

## 4.2 Inclined circular orbits.

We tried our method with the target point

$$x_B = (1\,\mathrm{s}, 6.3 \times 10^6\,\mathrm{m}, \pi/3\,\mathrm{rad}, -\pi/6\,\mathrm{rad}), \tag{31}$$

and a constellation of satellite obeying equations (26) with the following constants of motions:

$$\begin{aligned} t_{0,i} &= 0\,\mathrm{s} \quad \forall i, \\ r_{0,i} &= 4.2 \times 10^7\,\mathrm{m} \quad \forall i, \\ \phi_{0,1} = -50^\circ, \quad \phi_{0,2} &= 25^\circ, \quad \phi_{0,3} = 60^\circ, \quad \phi_{0,4} = -35^\circ, \end{aligned} \tag{32}$$

and the following inclination:

$$\begin{aligned} &\alpha_1 = \alpha_2 = 0^\circ, \quad \alpha_3 = \alpha_4 = 90^\circ, \\ &\beta_i = \gamma_i = 0^\circ \quad \forall i. \end{aligned} \tag{33}$$

The setting is illustrated in figure 5. We would like to remark that we used right-angle inclinations, and on the $x$-axis alone, for simplicity only, and we insist on the fact that the method can take on any inclination.

The fourth-order proper times obtained were :

| Sat. | $\tau$ [s] |
|---|---|
| 1 | 0.876400292365608987418752216077 |
| 2 | 0.869064153445191655337142565551 |
| 3 | 0.876261510308943339886614607209 |
| 4 | 0.865316554783636839167118613520 |

We now present the coordinates obtained for the target point. Time of reception (expected answer : $t_B = 1$ second):

| Order | $t_B$ [s] | Error [s] |
|---|---|---|
| 0 | 1.000004726525898470419881664511 | $4.7265 \times 10^{-6}$ |
| 0.5 | 1.000000000317128497477299209190 | $3.1713 \times 10^{-10}$ |
| 1 | 1.000000000000068448219616118322 | $6.8448 \times 10^{-14}$ |
| 1.5 | 1.000000000000000004276567159170 | $4.2766 \times 10^{-18}$ |
| 2 | 1.000000000000000000000289690092 | $2.8969 \times 10^{-22}$ |
| 2.5 | 0.999999999999999999999999890100 | $1.0990 \times 10^{-25}$ |
| 3 | 0.999999999999999999999999999967 | $3.3013 \times 10^{-29}$ |

Radius (expected answer : $r_B = 6.3 \times 10^6$ metres) :

| Order | $r_B$ [$10^6$ m] | Error |
|---|---|---|
| 0 | 6.297991095761440172150549624619 | $3.1887 \times 10^{-4}$ |
| 0.5 | 6.299999628817188546413163437225 | $5.8918 \times 10^{-8}$ |
| 1 | 6.29999999995439596529765837974 | $7.2387 \times 10^{-12}$ |
| 1.5 | 6.299999999999990124720349620199 | $1.5675 \times 10^{-15}$ |
| 2 | 6.299999999999999998562785122411 | $2.2813 \times 10^{-19}$ |
| 2.5 | 6.299999999999999999999734428831 | $4.2154 \times 10^{-23}$ |
| 3 | 6.299999999999999999999999941663 | $9.2599 \times 10^{-27}$ |

Polar angle (expected answer : $\theta_B = \pi/3$ radians) :

| Order | $\theta_B$ [rad] | Error |
|---|---|---|
| 0 | 1.047149706632208449047833646793 | $04.5688 \times 10^{-5}$ |
| 0.5 | 1.047197570905738944132225900057 | $1.8821 \times 10^{-8}$ |
| 1 | 1.047197551194164173407845979193 | $2.3239 \times 10^{-12}$ |
| 1.5 | 1.047197551196597732537859875917 | $1.3003 \times 10^{-17}$ |
| 2 | 1.047197551196597746256002605638 | $9.7201 \times 10^{-20}$ |
| 2.5 | 1.047197551196597746154180690784 | $3.2248 \times 10^{-23}$ |
| 3 | 1.047197551196597746154214468106 | $6.6967 \times 10^{-27}$ |

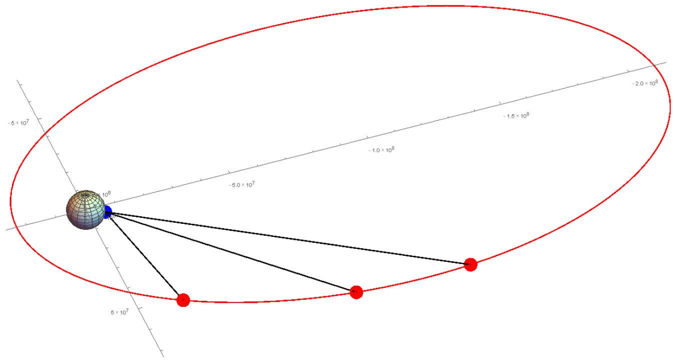

*Figure 3: Setting used to test the position solutions with equatorial elliptical orbits. The blue dot represents the target point B. The red dots and the red circle respectively represent the three satellites and their common orbit.*

Azimuthal angle (expected answer : $\phi_B = -\pi/6$ radians) :

| Order | $\phi_B$ [rad] | Error |
|---|---|---|
| 0 | -0.523783708168425960962291267763 | $3.5319 \times 10^{-4}$ |
| 0.5 | -0.523598750097320206425902671116 | $4.8703 \times 10^{-8}$ |
| 1 | -0.523598775603334562748029975696 | $9.6175 \times 10^{-12}$ |
| 1.5 | -0.523598775598297991009331092419 | $1.6846 \times 10^{-15}$ |
| 2 | -0.523598775598298873208007857222 | $2.5000 \times 10^{-19}$ |
| 2.5 | -0.523598775598298873077093409282 | $2.6397 \times 10^{-23}$ |
| 3 | -0.523598775598298873077107230492 | $1.0518 \times 10^{-28}$ |

The convergence is manifest for all four coordinates, and the relative errors is about $10^{-27}$ for each of them. As with the previous case, we felt that the third-order results already had a satisfying precision.

## 4.3 Equatorial elliptical orbits.

The target point here was chosen to be

$$x_B = (1\,\mathrm{s}, 6.3 \times 10^6\,\mathrm{m}, \pi/2\,\mathrm{rad}, 5\pi/6\,\mathrm{rad}), \tag{34}$$

and the orbital parameters of our satellites

$$\begin{aligned} t_{0,1} &= 15000\,\mathrm{s} \quad t_{0,2} = 35000\,\mathrm{s}, \quad t_{0,3} = 55000\,\mathrm{s}, \\ r_{0,i} &= 4.2 \times 10^7 \quad \forall i, \\ e_i &= 0.8 \forall i \end{aligned} \tag{35}$$

a situation pictured in figure 5.

As previously, we first simulated what would the satellites' broadcast proper times be, up to the fourth order. As explained above, we will be actually using the their eccentric anomaly

$\psi$ instead, which is obtained by numerically solving the last equation of (24) for $\psi$. We thus present the simulated $\psi$'s as well :

| Sat. | $\tau$ [s] | $\psi$ [rad] |
|---|---|---|
| 1 | 15000.836251849759120473648023015894 | 0.825661782110888876960384320820 |
| 2 | 35000.704684334625014682815831065773 | 1.331789462387076924703787399154 |
| 3 | 55000.601698883874606613218612358615 | 1.667648060084460111622883607577 |

We now present the coordinates obtained for the target point. Time of reception (expected answer : $t_B = 1$ second) :

| Order | $t_P$ [s] | Error |
|---|---|---|
| 0 | 0.999981147292624478443271196316 | $1.8852 \times 10^{-4}$ |
| 1 | 1.000000000311447240513143137083 | $3.1144 \times 10^{-10}$ |
| 2 | 0.999999999999991902940522415661 | $8.0971 \times 10^{-15}$ |
| 3 | 1.000000000000000000240528310966 | $2.4052 \times 10^{-19}$ |

Radius (expected answer : $r_P = 6.3 \times 10^6$ metres) :

| Order | $r_P$ [$10^6$ m] | Error |
|---|---|---|
| 0 | 6.301820859016794837474852781689 | $2.8902 \times 10^{-3}$ |
| 1 | 6.299998869872663870009048889219 | $1.7938 \times 10^{-7}$ |
| 2 | 6.299999999725109007327543474487 | $4.3633 \times 10^{-11}$ |
| 3 | 6.300000000000019882419173483962 | $3.1559 \times 10^{-15}$ |

Azimuthal angle (expected answer : $\phi_P = 5\pi/6$ radians) :

| Order | $\phi_P$ [rad] | Error |
|---|---|---|
| 0 | 2.617433575705034511049060737423 | $2.1401 \times 10^{-3}$ |
| 1 | 2.617993718180396219341713036414 | $6.1043 \times 10^{-8}$ |
| 2 | 2.617993878016665537549958414726 | $9.6147 \times 10^{-12}$ |
| 3 | 2.617993877991538123347549284001 | $1.6714 \times 10^{-14}$ |

The convergence is steady just like for the circular cases. We will note the absence of half-integer order in the elliptical formulae expansions. The absolute error for the radius is $1.99 \times 10^{-8}$ metres, notably more than with the circular case. We haven't performed a detailed analysis of the error in this proof-of-concepts study but the difference might simply arise from the fact that the satellites are much more distant from the target point on their highly eccentric orbit. Again we felt that this precision was satisfying enough not to carry on the method up to the fourth order.

# 5 Conclusion

Working with the post-Minkowskian approximation of general relativity, we provided an expanded formula for the time transfer function up to the fourth order.

We then sought to use the time transfer function to find analytical solution to the problem of position determination in a relativistic positioning system. We were able to find such solutions for the four coordinates of the receiving end, and that for satellites on both elliptic and circular orbits. With elliptical orbits, a basic numerical solving algorithm must still be used to convert the proper time (broadcast by the satellites) into the eccentric anomaly (used by the equations of motions). Our solutions come in the form of expansion in powers of $r_S/r_0$.

We tested our solutions on three different simulated satellite constellations. By computing the proper times up to the fourth order and the position up to the third, were able to correctly retrieve the position of the receiving end with an accuracy of 20 nanometres in the worst-case scenario.

## 5.1 Perspectives

The next step of this work would be to implement the Autonomous Basis of Coordinates [2][8], which would allow the satellite to locate themselves (thus creating a really autonomous navigation system) instead of being located though ground stations. This was done numerically in [8] but we believe an analytical treatment is possible.

Another point of interest to look at in the future would be the departures from the very simple Schwarzschild spacetime we considered. Gravitational perturbations such as the Earth's multipole or the influence of nearby bodies such as the Sun or the Moon do have a significant influence on the satellite's dynamics [3]. We believe that in this case, fourth-order calculations will be necessary to achieve a satisfying precision in the positioning.

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